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ScienceDirect
Journal of Applied Mathematics and Mechanics 71 (2007) 258–268
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Tollmien–Schlichting waves in a transonic boundary layer Excitation from the outer flow and from the surface☆

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Received 20 December 2005

Abstract

Perturbations in a boundary layer when there is unsteady free viscous-inviscid interaction at transonic velocities are investigated using a modified "triple-deck" model. Two problems are considered: the effect of acoustic perturbations arriving from outside and of perturbations excited from a streamline surface with a vibrator situated on it. The modification of the model consists of taking into account singular terms of the transonic expansion, which enables the Lin–Reissner–Tsien equation to be improved and enables non-stationary and non-linear phenomena in the flow to be described more correctly. It is shown that the modified model enables additional perturbations, overlooked when using the classical triple-deck model, to be taken into account.

Investigation of the development in a boundary layer of small perturbations generated by different sources represents a range of problems of the theory of boundlary-layer susceptibility. The growth of perturbations may be one of the reasons for loss of stability by a boundary layer, and may later lead to separation of the boundary layer. There is a voluminous literature available at the present time devoted to an investigation of the susceptibility of the boundary layer at subsonic velocities. A review of the results obtained can be found in Ref. 1.

One of the reasons for the occurrence of small perturbations in a boundary layer (Tollmien–Schlichting waves) may be the work of the vibrator placed on it. Problems of this kind previously considered cover the case of the interaction of the boundary layer with steady supersonic[2,3] and steady subsonic[4] flows. For supersonic interaction the asymptotic form of the pressure was determined both at infinity and for long values of the time at fixed operation frequencies of the vibrator and specified vibrator geometry. The results obtained showed that the pressure perturbations are concentrated around the vibrator and decay rapidly with distance from it, irrespective of the frequency. When the frequency increases the amplitude of the pressure perturbation increases. Note that the pressure is also perturbed upstream from the point where the vibrator is situated. The perturbations introduced by the vibrator decay more rapidly than the perturbations from a stationary obstacle with the same geometry. In the case of supersonic flow, there is a critical vibrator operating frequency. When it operates at a subcritical frequency the perturbations in the boundary layer decay, at the critical frequency they do not decay, while at supercritical frequencies they grow.

Transonic flow differs considerably both from subsonic and supersonic flows: the features of supersonic flow (primarily, shockwaves) are associated with the possibility of an increase in weak perturbations of the boundary layer,

☆ *Prikl. Mat. Mekh.* Vol. 71, No. 2, pp. 289–300, 2007.
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0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2007.06.009

discovered in Ref. 5, irrespective of whether the velocity of such flow exceeds the velocity of sound or not (for purely supersonic flow no growing perturbations are observed[2]).

When modelling flows with interaction in the transonic velocity range the so-called Neiland-Stewartson triple-deck model is used,[6,7] which was subsequently employed to investigate the interaction under unsteady conditions also.[5,8] An investigation of the free viscous-non-viscous interaction at transonic velocities enabled considerable progress to be made in solving many problems (for a review of research see Ref. 9).

In the triple-deck model of transonic unsteady interaction,[8] the external (non-viscous unsteady) flow is described by the Lin–Reissner–Tsien (LRT) equation for the potential of the perturbed velocity. This equation, as is well known,[10] has a number of drawbacks, which do not enable the propagation in the flow of unsteady perturbations to be correctly described. A modified triple-deck model for investigating stability was proposed in Ref. 11. The modification of the model consisted of using a modified LRT equation, containing an additional singular term and which describes, unlike the classical LRT equation,[12] the propagation of unsteady perturbations in any direction in the flow field (similar to the Euler system of equations). The use of this model to investigate problems of unsteady free viscous-non-viscous interaction at transonic velocities enable the features of the development of unsteady perturbations to be described more accurately.[11] It is precisely these (the perturbations ignored by the LRT equation) which generate additional perturbations of the boundary layer (ignored in the usual triple-deck model[5]) which may grow under certain conditions.

The investigation, based on the modified model,[11] of the problem of a vibrator placed in the boundary layer with interaction, given below, has enabled us to detect perturbations excited by the vibrator, which have not been described previously.

Perturbations in the boundary layer may also be caused by external factors. It was shown in Ref. 13, on the basis of the theory of unsteady transonic free interaction, that arriving acoustic perturbations may generate perturbations in the boundary layer, and a reference was made to the results obtained in Ref. 14 for subsonic velocities. As was pointed out above, transonic flow exhibits considerable differences compared with flow at both subsonic and supersonic velocities. Unlike subsonic flow[13] Tollmien–Schlichting waves, when there is free interaction under transonic conditions, arose under smooth boundary conditions (there were no bulges, kinks, projections, etc. on the surfaces which limited the flow). Under subsonic conditions, the presence of such sigularities is necessary in order to generate perturbations,[14] which particularly emphasises the non-linearity of the process by which these oscillations are excited.

Below, when investigating the conversion of external acoustic perturbations in Tollmien–Schlichting waves, we propose to use a modified triple-deck model of unsteady transonic interaction.[11] It is shown that additional perturbations (taken into account by the modified LRT equation in the outer flow) generate an additional wave packet in the boundary layer.

1. The problem of a vibrator in a transonic boundary layer

When a vibrator is placed on a solid surface $y = y_w$ in the transonic flow, according to the no-slip and impermeability boundary conditions we have the following longitudinal and transverse components of the flow velocity

$$u(t, x, y_w) = 0, \quad v(t, x, y_w) = y_{wt} \tag{1.1}$$

We will assume[2] that the vibrator performs harmonic oscillations with an amplitude characterized by the parameter $\tilde{\delta}$, i.e.

$$y_w = \begin{cases} \tilde{\delta} f_w(t, x) = \tilde{\delta} f_1 \sin \omega_0 t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

The function f_1 specifies the particular geometry of the vibrator.

We will use the standard expansions[5] for the flow parameters in the boundary layer on a solid surface. In this case, the equations describing the changes in the principal terms of the expansion have the usual form of the equations of an unsteady boundary layer for an incompressible fluid

$$u_x + v_y = 0, \quad p_y = 0, \quad u_t + uu_x + vu_y = -p_x + u_{yy} \tag{1.2}$$

The feature of the triple-deck model is that a certain transition flow – a “middle deck”, is distinguished in the region being investigated, which serves to match the boundary layer with the outer inviscid potential flow. The equations which model the transition flow were derived and integrated in relation to the corresponding boundary conditions in Refs. 5,8.Here we will only present the following parts of the overall analysis of the matching conditions

$$\phi_x(t,x,0) = -p(t,x), \quad \phi_{y_1}(t,x,0) = -A_x(t,x) \text{ as } y_1 \to 0 \tag{1.3}$$

$$u \to y + A(t,x) \text{ as } y \to \infty \tag{1.4}$$

($\varphi(t, x, y)$ is the velocity potential and $A(t, x)$ is a function of the instantaneous displacement of the streamlines of the transition flow). Note that, on transferring to the three-deck model the spatial coordinates and the time are transformed, but the x coordinate and the time t are the same for all the decks, and the y_1 and y coordinates have different scales, since the boundary layer thickness and the thickness of the transition flow are different.

Inviscid transonic flow far from the plate can be assumed to be approximately vortex-free. To describe it we will use the modified linear Lin-Reissner-Tsien (LRT) equation[11]

$$\delta\phi_{tt} + 2\phi_{xt} + K_\infty\phi_{xx} - \phi_{y_1y_1} = 0, \quad \delta \ll 1, \quad K_\infty = (M_\infty^2 - 1)/\delta \tag{1.5}$$

The term with the second derivative with respect to time in this equation is singular. This term is omitted in the usual LRT equation, but in this case the model of the flow being investigated only describes unsteady perturbations propagating upstream.

To fix our ideas, we will choose Blasius flow as the main flow. The initial conditions have the form

$$t = 0: u = y, \quad \upsilon = -y_{wt}, \quad p = \text{const}$$

(there are no perturbations before the vibrator begins to operate). The boundary condition is

$$x \to -\infty: u \to y, \ \upsilon \to 0, \ p \to \text{const}$$

(there are no arriving perturbations).

We will assume that the amplitude of the vibrator oscillations are small: $\tilde{\delta} \to 0$. Representing the parameters of the perturbed flow in the form of series in powers of $\tilde{\delta}$

$$u = y + \tilde{\delta}u_e + \ldots, \quad \upsilon = \tilde{\delta}\upsilon_e + \ldots, \quad p = \tilde{\delta}p_e + \ldots$$

and substituting them into system (1.2), we have

$$u_{e_x} + \upsilon_{e_y} = 0, \quad p_{e_y} = 0, \quad u_{e_t} + yu_{e_x} + \upsilon_e = -p_{e_x} + u_{e_{yy}} \tag{1.6}$$

The chosen expansions hold when $y \gg \tilde{\delta}\, u_e$ (at a sufficient distance from the solid boundary). In the region close to the wall we must introduce[2] an additional subregion with parameters

$$u = \tilde{\delta}u_{eb} + \ldots, \quad \upsilon = \tilde{\delta}\upsilon_{eb} + \ldots, \quad y_b = y\tilde{\delta}^{-1} \tag{1.7}$$

The matching conditions on the boundary of this subregion and the boundary layer have the form

$$\tilde{\delta}u_{eb}(t,x,y_b) \to y + \tilde{\delta}u_e(t,x,0) + \ldots, \quad \tilde{\delta}\upsilon_{eb}(t,x,y_b) \to \tilde{\delta}\upsilon_e(t,x,0) + \ldots \tag{1.8}$$

On the surface of the vibrator, with $y = y_w$, by conditions (1.1) we have

$$y = y_w : u_{eb}(t,x,f_1(x)\sin(\omega_0 t)) = 0, \ \upsilon_{eb}(t,x,f_1(x)\sin(\omega_0 t)) = \omega_0 f_1(x)\cos(\omega_0 t) \tag{1.9}$$

Substituting expansions (1.7) into system (1.6) we obtain

$$\upsilon_{eb_y} = 0, \quad u_{eb_{yy}} = 0 \tag{1.10}$$

The solution of system (1.10) with boundary conditions (1.9) has the form

$$u_{eb} = \lfloor y_b - f_1(x)\sin(\omega_0 t)\rfloor \text{const}(t,x), \quad \upsilon_{eb} = \omega_0 f_1(x)\cos(\omega_0 t)$$

The matching condition (1.8) gives const$(t, x)\equiv 1$ and

$$u_e(t,x,0) = -f_1(x)\sin(\omega_0 t), \quad \upsilon_e(t,x,0) = \omega_0 f_1(x)\cos(\omega_0 t) \tag{1.11}$$

From the initial conditions and the boundary conditions as $x \to -\infty$ and $y \to \infty$ (condition (1.4)) we obtain

$$u_e(0, x, y) = 0; \quad u_e \to 0 \text{ as } x \to -\infty; \quad u_e \to A \text{ as } y \to \infty \tag{1.12}$$

The solution of linear system (1.6), which satisfies conditions (1.11) and (1.12), can be sought using a Fourier transformation with respect to x and a Laplace transformation with respect to t. For the transform u_e we have

$$\bar{u}_e = \int_{-\infty}^{\infty} dx \int_0^{\infty} u_e(t, x, y)\exp(-\omega_e t - ikx)dt, \quad \omega_e = i\omega$$

$$\bar{u}_e(\omega_e, k, 0) = -\overline{f_w}(\omega_e, k) = 2k^{-2}\omega_0(\omega_0^2 + \omega_e^2)^{-1}\overline{f_0}$$

The quantity $\bar{f}_0$ is determined by the vibrator geometry.

Carrying out these transformations, system (1.6) can be reduced to a single equation by differentiating the last equation of system (1.6) with respect to y and eliminating $\bar{\upsilon}_{e_y}$ using the first equation of system (1.6). We obtain

$$\frac{d^3\bar{u}_e}{dy^3} - (\omega_e + iky)\frac{d\bar{u}_e}{dy} = 0 \tag{1.13}$$

To solve Eq. (1.13) we need three boundary conditions. The first boundary condition can be obtained by Fourier and Laplace transformations of the first condition of (1.11)

$$\bar{u}_e = -\overline{f_w}, \text{ when } y = 0 \tag{1.14}$$

The second boundary condition is derived as follows. As a result of Fourier and Laplace transformations the last equation of (1.6) gives

$$-(\omega_e + iky)\bar{u}_e + \bar{\upsilon}_e = \frac{d^2\bar{u}_e}{dy^2} + ik\bar{p}_e \tag{1.15}$$

Calculations show that

$$\bar{u}_e = \omega_0(\omega_e^2 + \omega_0^2)^{-1}\overline{f_1}, \quad \bar{\upsilon}_e = \omega_0\omega_e(\omega_e^2 + \omega_0^2)^{-1}\overline{f_1}$$

Substituting these expressions into Eq. (1.15), we have

$$\frac{d^2\bar{u}_e}{dy^2} = -ik\bar{p}_e \text{ when } y = 0 \tag{1.16}$$

The third boundary condition is the transformation of the last condition of (1.12)

$$\bar{u}_e \to \bar{A} \text{ as } y \to \infty \tag{1.17}$$

Introducing the new independent variable $z = (ik)^{-2/3}(\omega_e + iky)$, we can reduce Eq. (1.13) to an Airy equation[15] in $d\bar{u}_e/dz$. One of the solutions of Eq. (1.13) can be represented as follows:

$$\frac{d\bar{u}_e}{dz} = C_0\mathrm{Ai}(z) \tag{1.18}$$

where $\mathrm{Ai}(z)$ is the Airy function. The second linearly independent solution $\mathrm{Bi}(z)$ is an unbounded function as $z \to \infty$ and can be dropped.

In turn, the solution of Eq. (1.18) is

$$\bar{u}_e = C_0\int_{\Omega}^{z} \mathrm{Ai}(\xi)d\xi + C_1, \quad \Omega = (ik)^{-2/3}\omega_e \tag{1.19}$$

The boundary conditions (1.14) and (1.17), on changing to the new variable z, remain without change and are now satisfied when $z = \Omega$ and $z \to \infty$ respectively. Boundary condition (1.16) becomes

$$\frac{d^2\bar{u}_e}{dz^2} = -(ik)^{1/3}\bar{p}_e \quad \text{when} \quad z = \Omega \tag{1.20}$$

From condition (1.14), taking the above remarks into account, we have from Eq. (1.19)

$$C_1 = -\bar{f}_w$$

From condition (1.20) we obtain

$$C_0\frac{d\text{Ai}}{dz} - (ik)^{1/3}\bar{p}_e = 0, \quad z = \Omega \tag{1.21}$$

From condition (1.17) we have

$$C_0 I(\Omega) + C_1 - \bar{A} = 0, \quad z \to \infty; \quad I(z) = \int_z^\infty \text{Ai}(\xi)d\xi \tag{1.22}$$

For $\bar{p}_e$, $\bar{A}$, by carrying out Fourier and Laplace transformations of matching condition (1.3), we obtain

$$\bar{p}_e = ik\bar{\phi}, \quad d\bar{\phi}/dy_1 = ik\bar{A}$$

The transform $\bar{\phi}$ is found from the converted Eq. (1.5)

$$\frac{d^2\bar{\phi}}{dy_1^2} = R^2\bar{\phi}; \quad R^2 = \delta\omega_e^2 + 2ik\omega_e - k^2K_\infty \tag{1.23}$$

The solution of Eq. (1.23) has the form

$$\bar{\phi} = \bar{\phi}(0)\exp(\pm Ry_1)$$

We choose the minus sign in the solution obtained. This ensures that $\bar{\phi}$ decays as $y_1 \to \infty$, and taking it into account instead of Eqs. (1.21) and (1.22) we have

$$C_0\frac{d\text{Ai}}{dz} - (ik)^{4/3}\bar{\phi} = 0, \quad C_0 I(\Omega) + \bar{f}_w + R^*\bar{\phi} = 0; \quad R^* = (ik)^{-1}R$$

whence, eliminating $\bar{\phi}$, we find C_0 and obtain

$$\bar{u}_e = -\bar{f}_w\frac{I^*(z) + R^*\dfrac{d\text{Ai}}{dz}}{I^*(\Omega) - R^*\dfrac{d\text{Ai}}{dz}}; \quad I^*(z) = (ik)^{4/3}I(z) \tag{1.24}$$

From the solution of (1.24) we can obtain $\bar{p}_e$ and $\bar{v}_e$. The expression for $\bar{p}_e$ is simplest since it is independent of z. By condition (1.20) we have

$$p_e = -\frac{\omega_0}{2\pi^2}\int_{-\infty}^{\infty}\bar{f}_0 i^{1/3}k^{-2/3}J dk; \quad J = \int_{l-i\infty}^{l+i\infty}(\omega_0^2 + \omega_e^2)^{-1}\frac{\dfrac{d\text{Ai}}{dz}\exp(\omega_e t)}{I^*(\Omega) - R^*\dfrac{d\text{Ai}}{dz}}d\omega_e \tag{1.25}$$

As can be seen from expression (1.25), consideration of the unsteady processes in the outer transonic flow leads to a change in the form of the expression for the pressure compared with the solution obtained for the problem of a vibrator in the boundary layer with interaction at supersonic velocities under steady conditions.[2] The main difference is the presence of the factor R^* on $d\text{Ai}/dz$ in the denominator of the expression for J.

When $\omega_e = 0$ (steady flow) and $K_\infty = 1$ (emergence from the transonic range into the area of supersonic velocities) we find $R = ik$, and from expressison (1.25) we have

$$p_e = -\frac{\omega_0}{2\pi^2}\int_{-\infty}^{\infty} \bar{f}_0 i^{1/3} k^{-2/3} J dk \tag{1.26}$$

i.e. a result obtained previously when solving the problem of a vibrator at a supersonic velocity of the outer steady flow.[2]

The expression for the pressure obtained previously in Ref. 4 in the case of subsonic outer flow is a non-analytic function, since it contains $|k|$. Solution (1.25) converts into this solution when $\omega_e = 0$, $k > 0$, $K_\infty = -1$ (emergence from the transonic range into the region of subsonic velocities).

To evaluate the integral in expression (1.25), which defines p_e, and also to determine the other flow parameters, one can use applications of the theory of residues[16] (Cauchy's theorem), and the calculation of the definite integrals reduces to finding the residues of the integrand in the region with a suitably chosen boundary.

Consider the integral J occurring in expression (1.25). The residues of the integrand in J are determined by the zeros of its denominator, and it vanishes when $\omega_e = i\omega_0$, $\omega_e = -i\omega_0$, and also when

$$\frac{1}{I(\Omega)}\frac{d\mathrm{Ai}}{dz} = \frac{(ik)^{7/3}}{R} \tag{1.27}$$

Eq. (1.27) is a dispersion relation (it connects ω_e and k) for free oscillations in the boundary layer with unsteady interactions at transonic velocities.[11] This relation, with the term $\delta\omega_e^2$ in the integrand, differs from the dispersion relation obtained in Ref. 5 when considering the same problem (i.e. the problem of the stability of a boundary layer with unsteady interaction at transonic velocities with respect to weak perturbations of the form $f(y)\exp(\omega t + ikx)$) using the usual triple-deck model. To indicate this difference, it was suggested in Ref. 11 that relation (1.27) should be called the modified dispersion relation; the origin of this difference and the features of the flow arising from it were also explained in Ref. 11.

If we assume the outer flow to be steady (we neglect the dependence on ω_e in the first part of relation (1.27), putting $\omega_e = 0$), we obtain the following new dispersion relation from dispersion relation (1.27)

$$\frac{1}{I(\Omega)}\frac{d\mathrm{Ai}}{dz} = (ik)^{4/3} K_\infty^{-1/2} \tag{1.28}$$

When $K_\infty = 1$ (the model supersonic flow) dispersion relation (1.28) is identical with the dispersion relation obtained in Ref. 17 when investigating the stability of a boundary layer freely interacting with steady supersonic flow, and when $K_\infty = -1$ (model subsonic flow) and $k > 0$, dispersion relation (1.28) is identical with the dispersion relation obtained in Ref. 18 when investigating the stability of a boundary layer freely interacting with steady subsonic flow. It can be seen that additional roots of the dispersion relation are obtained when the unsteady nature of the outer flow is taken into account (see also Ref. 10).

An analytical investigation of dispersion relations (1.27) and (1.28) is quite difficult. The roots of dispersion relation (1.28) and $K_\infty = \pm 1$, and also of dispersion relation (1.27) when $\delta = 0$, were obtained numerically in Refs. 5,17. For these dispersion relations, asymptotic forms were also constructed for large values of Ω in Refs. 5,11, which occur when $\omega_e \to \infty$, $k \to \infty$ (the high-frequency approximation). The asymptotic form for small Ω was not constructed. Such an asymptotic form arises, as will be shown below, when $\omega_e \to 0$, $k \to 0$, when the ratio ω_e/k is finite.

The following procedure is used to find the roots of dispersion relations (1.27) and (1.28) numerically, as can be seen, when $k \to 0$ the right-hand side of dispersion relation (1.28) approaches zero. The left-hand side of both dispersion relations (1.27) and (1.28) is the same and contains an integral of the Airy function and the derivative of this function. The integral in the case considered is a finite non-zero quantity, while the derivative of the Airy function has an infinite number of zeros for negative values of the argument.[15] This fact (as shown in Ref. 19 when analysing dispersion relation (1.28) with $K_\infty = 1$) also determines the roots of dispersion relation (1.28) when $k \to 0$. The roots when $k \neq 0$ are found by Newton's iteration method, the initial data in the iteration process being the roots obtained when $k \to 0$. When using this method the problem of the branching of the roots is important.

As it turned out, in the case of interaction with a steady supersonic flow for real k, all the roots of dispersion relation (1.28) lie in the second quadrant (for $k < 0$) or the third quadrant (for $k > 0$) of the ω_e plane, which enables the contour of integration with respect to ω_e to be constructed for solution (1.26) as follows[17]: we draw two rays C_1 and C_2, which make angles of $\pi/2+\alpha$ and $-\pi/2-\alpha$, $0<\alpha<0.07\pi$ with the abscissa axis of the ω_e plane, and we connect these rays by arcs of fairly large radius with the abscissa axis of the ω_e plane. Here the residues at the points $\omega_e = i\omega_0$, $\omega_e = -i\omega_0$ turn out to be inside the chosen region, whereas the residues corresponding to the roots (1.28), are outside it and will not participate in the evaluation of the integral in solution (1.26).

A specific feature of dispersion relation (1.27) is the fact that when $k>0$, in addition to the roots determined by the zeros of the derivative $d\mathrm{Ai}/dz$, there are zeros determined by the expression under the radical sign on the right-hand side of relation (1.27).

Since, when $k\to 0$, the integral $I(\Omega)$ does not significantly determine the roots of the dispersion relation, we will put it equal to the limit value 1/3 when $\Omega\to 0$. Dispersion relation (1.27) then becomes

$$3R\frac{d\mathrm{Ai}}{dz} = (ik)^{7/3}$$

As can be seen, the roots defined by the expression under the radical sign R, when $k\to 0$ give the asymptotic relation $\omega_e \sim k$. In this case, dispersion relation (1.27), in addition to the roots defined by the zeros of the derivative $d\mathrm{Ai}/dz$, has the additional roots

$$\omega_e = k(-i \pm \sqrt{K_\infty\delta - 1})\delta^{-1} \qquad (1.29)$$

These are small quantities as $k\to 0$.

Eq. (1.29) has two roots. For real positive k the form of the roots is determined by the sign and the value of K_∞. When $K_\infty > \delta^{-1}$ (supersonic flow at high Mach numbers), ω_e has real and imaginary parts and there is always one growing and one decaying root. When $K_\infty < \delta^{-1}$, including also when $K_\infty < 0$ (subsonic flow), its solution for ω_e is pure imaginary. There is a neutral frequency $K_\infty = \delta^{-1}$.

When $\delta = 0$ one of the roots of Eq. (1.29) drops out and only one additional root of dispersion relation (1.27) remains of the form

$$\omega_e = -ikK_\infty/2 \qquad (1.30)$$

which gives the relation $\omega_e \sim k$ when $|K_\infty| \sim 1$. For real positive k the root (1.30) is pure imaginary. Travelling-wave type solutions correspond to this root. When $K_\infty > 0$ (the outer flow is supersonic) we have $\omega_e < 0$, and the waves travel downstream. When $K_\infty < 0$ (the outer flow is subsonic), correspondingly $\omega > 0$ and the waves travel upstream.

Hence, the results obtained show that when the vibrator operates in the boundary layer, freely interacting with the unsteady inviscid flow at transonic velocities, there are perturbations, not taken into account by previously used models. Under certain conditions these perturbations can grow and have a considerable affect on the flow field that arises under these conditions.

2. The generation of Tollmien–Schlichting waves by sound

Consider constant flow around a plane plate, oriented in the direction of the flow velocity.[13] We will assume that the velocity of the flow, unperturbed by the plate, is close to the velocity of sound.

As in Section 1, we will use the triple-deck model, including the modified linear Lin-Reissner-Tsien Eq. (1.5), and the equations of an unsteady boundary layer in an incompressible fluid (1.2). We will take the usual no-slip and impermeability conditions on the plate surface

$$u = v = 0 \quad \text{when} \quad y = 0$$

The boundary layer on the plate surface and the flow in the outer region are joined by the transition-flow conditions (1.3) and (1.4). The perturbations, as one moves away from the flow interaction region, will be assumed to be bounded. We will specify once again the condition for the arrival of acoustic perturbations (the initial conditions are $t=0$)

$$\phi(0, x, y_1) = \phi_0(x, y_1)$$

Consider the behaviour on the plate surface of small perturbations, which we will assume to be additions to the parameters of a certain background (Blasius flow)

$$u = y + \tilde{\delta}u_1 + \dots, \quad v = \tilde{\delta}v_1 + \dots, \quad p = p_\infty + \tilde{\delta}p_1 + \dots$$

Assuming the amplitude $\tilde{\delta}$ of such perturbations to be small ($\tilde{\delta} \to 0$), we will linearize the equations of the above model with respect to $\tilde{\delta}$. We will obtain a solution of the linear system of equations obtained by using a Fourier transformation with respect to the x coordinate and a Laplace transformation with respect to t. When carrying out the Laplace transformation in this problem, it is necessary to take the initial data into account, so that

$$\frac{\partial \bar{u}_1}{\partial t} \to -\omega\tilde{u} + u(0, x, y)$$

Just as in Section 1, the solution of the linear system of ordinary differential equations obtained can be reduced to a single equation

$$\frac{d^3\bar{u}_1}{dy^3} - (\omega + iky)\frac{d\bar{u}_1}{dy} = \frac{d\tilde{u}}{dy}(0, k, y); \quad \tilde{u}(0, k, y) = \int u(0, \xi, y)\exp(-ik\xi)d\xi \tag{2.1}$$

By replacing the independent variable $z = (ik)^{-2/3}(\omega + iky)$, Eq. (2.1) can be reduced to an equation, the homogeneous part of which is the Airy equation in the derivative $d\bar{u}_1/dz$. The solution of the homogeneous equation is easily obtained in terms of the Airy function[15] (see Eq. (1.20)), dropping the second linearly independent solution of this equation as being unbounded as $z \to \infty$. The function $\bar{u}_1$ is then obtained by quadratures, in the same way as was done in Section 1. Now, when Eq. (2.1) is inhomogeneous, the required solution has a somewhat different form.

Like Eq. (2.1), the inhomogeneous equation was solved previously in Ref. 14, and the solution was chosen in the form

$$\frac{d\bar{u}_1}{dz} = C_0(\omega, k)\mathrm{Ai}(z) + C_1\varphi(z)$$

where

$$\frac{d^2\varphi}{dz^2} - z\varphi = g, \quad \varphi(\Omega) = \varphi(\infty) = 0 \tag{2.2}$$

where g is a specified function, representing the roughness of the surface in the flow. Eq. (2.2) describes the leading terms of the expansions of the required parameters (when investigating the generation of Tollmien–Schlichting waves by sound in a boundary layer, freely interacting with a steady subsonic flow), and also explains its non-uniformity. Note that the solution of Eq. (2.2) is much simpler than the solution of Eq. (2.1).

Since the fundamental system of solutions of the homogeneous equation corresponding to Eq. (2.1) is known, the solution of Eq. (2.1) can be represented in the form[20]

$$\frac{d\bar{u}_1}{dz} = C_0\mathrm{Ai}(z) + (ik)^{1/3}Q(z, \Omega)$$

$$Q(z, \Omega) = \mathrm{Bi}(z)\int_\Omega^z \frac{\mathrm{Ai}(\xi)}{W(\xi)}\frac{d\tilde{u}}{d\xi}d\xi - \mathrm{Ai}(z)\int_\Omega^z \frac{\mathrm{Bi}(\xi)}{W(\xi)}\frac{d\tilde{u}}{d\xi}d\xi, \quad W(z) = \mathrm{Ai}(z)\frac{d\mathrm{Bi}}{dz} - \mathrm{Bi}(z)\frac{d\mathrm{Ai}}{dz}$$

whence it follows that

$$\bar{u}_1 = C_0I(\Omega) + (ik)^{1/3}\int_\Omega^z Q(\xi, \Omega)d\xi + C_1$$

To determine the constants C_0 and C_1 we will use the corresponding boundary conditions of the triple-deck model after they have been linearized and after a Fourier transformation with respect to the x coordinate and a Laplace transformation with respect to t. From the transformed no-slip condition we have $\bar{u}_1 = 0$ when $z = \Omega$; consequently, $C_1 = 0$. From the converted matching condition with the transition flow (1.12) $\bar{u} \to \bar{A}(\omega, k)$ as $y \to \infty$, we have

$$\bar{A} = C_0 I(\Omega) + (ik)^{1/3} \int_{\Omega}^{\infty} Q(\xi, \Omega) d\xi$$

The bounded solution of the linearized modified Lin-Reissner-Tsien Eq. (1.5), transformed using the Fourier and Laplace transformations, has the form

$$\frac{d^2\bar{\phi}}{dy_1^2} - R^2\bar{\phi} = \delta[\tilde{\phi}(0, k, y_1)\omega + \tilde{\phi}_t(0, k, y_1)] + 2ik\tilde{\phi}(0, k, y_1) \tag{2.3}$$

Eq. (2.3), unlike the similar equation in the vibrator problem (see Eq. (1.23)) is inhomogeneous: its right-hand side describes the effect of the acoustic perturbations arriving from outside, and $\tilde{\phi}(0, k, y_1)$, $\tilde{\phi}_t(0, k, y_1)$ are the Fourier-transformed values of $\phi(t, x, y_1)$, $\phi_t(t, x, y_1)$ when $t = 0$; for example, $\tilde{\phi}(0, k, y_1) = \int \phi(0, \xi, y_1) \exp(-ik\xi) d\xi$.

Ignoring the effect of external perturbations, Eq. (2.3) is homogeneous and has the solution

$$\bar{\phi}(y_1) = \bar{\phi}_{01}(0)\exp(-Ry_1) + \bar{\phi}_{02}(0)\exp(Ry_1)$$

in which we put $\bar{\phi}_{02}(0) = 0$, in order to ensure that $\bar{\phi}$ is bounded as $y_1 \to \infty$. Solution (2.3) can then be written in the form (see Ref. 20)

$$\bar{\phi}(y_1) = \bar{\phi}_{01}(0)\exp(-Ry_1) + \int_0^{y_1} h[\exp(R(\xi - y_1)) + \exp(-R(\xi - y_1))] d\xi$$

$$h = \delta[\tilde{\phi}(0, k, y_1)\omega + \tilde{\phi}_t(0, k, y_1)] + 2ik\tilde{\phi}(0, k, y_1)$$

According to the transformed matching conditions (1.3)

$$d\bar{\phi}/dy_1 = ik\bar{A}, \quad \bar{p}_1 = ik\bar{\phi}, \quad y_1 \to 0$$

we obtain

$$ik\bar{A} = -R\bar{\phi} + h$$

Taking into account the relation

$$\bar{p}_1 = -(ik)^{-1}\frac{d^2\bar{u}_1}{dy^2} = -(ik)^{-1/3}\frac{d^2\bar{u}_1}{dz^2}, \quad \text{or} \quad \bar{p}_1 = -C_0(ik)^{-1/3}\frac{d\text{Ai}}{dz}$$

we have

$$ik\bar{\phi} = -C_0(ik)^{-1/3}\frac{d\text{Ai}}{dz}$$

Eliminating $\bar{\phi}$, we obtain

$$ik\bar{A} = RC_0(ik)^{-4/3}\frac{d\text{Ai}}{dz} + h$$

Then, eliminating $\bar{A}$, we determine C_0. The solution is then constructed with the C_0 obtained; we give below (as the simplest, independent of z) expression for the transform of the pressure

$$\bar{p}_1 = \frac{(ik)^{5/3}\int_{\Omega}^{\infty} Q(\xi, \Omega) d\xi - \delta[\tilde{\phi}(0, k)\omega + \tilde{\phi}_t(0, k)] - 2(ik)^{1/3}\tilde{\phi}(0, k)}{R^*\frac{d\text{Ai}}{dz} - I^*(\Omega)} \frac{d\text{Ai}}{dz} \tag{2.4}$$

When $\delta = 0$ expression (2.4) takes the form

$$\bar{p}_1 = -k^2 \frac{\frac{d\mathrm{Ai}}{dz}\int_0^\infty \bar{\phi}_0(k,\xi)d\xi}{R_0\frac{d\mathrm{Ai}}{dz} - (ik)^{7/3}I(\Omega)}; \quad R_0^2 = 2ik\omega - k^2K_\infty \tag{2.5}$$

and was derived previously in Ref. 13.

When investigating the interaction in the subsonic state the following was obtained[14]

$$\bar{p}_1 = \bar{f}(k)k\frac{\frac{d\mathrm{Ai}}{dz}\int_0^\infty \varphi(\omega,k,\xi)d\xi - I(\Omega)\frac{d\varphi}{dz}}{\frac{d\mathrm{Ai}}{dz} - (ik)^{1/3}|k|I(\Omega)} \tag{2.6}$$

where $\bar{f}$ is the Fourier-transformed function, defining the boundary of the roughness on the wall around which the flow occurs $y = f(x)$.[14] The expression in the denominator of solution (2.5) becomes the denominator of solution (2.6) when $\omega = 0$, $K_\infty = -1$ (steady model subsonic flow). In the numerator of solution (2.6) there is an additional term, which depends on $d\varphi/dz$. Obviously, the main difference between solution (2.6) and solutions (2.4) and (2.5) is the fact that in solution (2.6) the transformed form of the roughness of the wall in the flow enters as a factor, i.e. in a subsonic boundary layer with smooth boundary conditions there are no pressure perturbations.

To determine the parameters of the flow being investigated we must invert the Fourier and Laplace transformations of the expressions obtained for the transforms, in particular, solution (2.4). One of the methods of finding the integrals obtained by such inversion is to use the theorem of residues.[16] In this case, the expression for the flow parameters, for example, the pressure, can be represented in the form of a series, the terms of which are calculated from the zeros of the denominator of expression (2.4). Equating the denominator of expression (2.4) to zero we obtain dispersion relation (1.27).

Dispersion relation (1.27) has been investigated in detail for $\delta = 0$ in Ref. 5 and in the general case in Ref. 11. It was shown that it has one increasing root (when $\delta = 0$) or two increasing roots (when $\delta \neq 0$) and a denumerable number of decreasing roots. The increasing roots are (only principal terms in k are retained)

$$\omega_1 = -ik/\delta + \sqrt{ik/2}(1-i), \quad \omega_2 = (-i+\sqrt{3})k^{5/3}/2$$

It can be seen that when $\delta = 0$ the first of these roots drops out. The second root corresponds to that obtained previously in Ref. 5.

In Ref. 13 the expression for the pressure was represented in the form

$$p_1 = \mathrm{Real}\left\{p^0(t,x) + \sum_{n=1}^{\infty} p^{(n)}(t,x)\right\} \tag{2.7}$$

where $p^0(t, x)$, $p^{(n)}(t, x)$ corresponded to the roots obtained for the dispersion relation. An estimate of the value and an investigation of the asymptotic behaviour of the terms of the series (2.7) (carried out earlier in Ref. 13 for $\delta = 0$) showed that as $t \to \infty$ the pressure is mainly determined by the contribution of one (increasing) root. Numerical calculations showed[13] that the pressure distribution clearly exhibits the form of a wave packet.

When $\delta \neq 0$ there are two increasing roots; consequently two wave packets, defined by these roots, will propagate in the flow field.

Acknowledgements

This research was supported financially by the Russian Foundation for Basic Research (04-01-00807) and the Programme for the Support of the Leading Scientific Schools (NSh-6791.2006.1).

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Translated by R.C.G.